Improved Numerical Method for Unsteady Lifting Surfaces in Incompressible Flow

A. Ichikawa* and S. Ando†
Nagoya University, Nagoya, Japan

An improved numerical method has been developed for unsteady thin rectangular wings in incompressible flow. To satisfy the wing boundary conditions, the spanwise pressure distribution is assumed to be stepwise constant, while the chordwise integral is reduced to a finite sum using several kinds of quadratures; Cauchy and logarithmic singularities are treated appropriately. The present method gives excellent results over a wide range of reduced frequency *k* (*k* = 0.0-5.0) within reasonable computer time. Extension to compressible flow and to swept tapered wings applications is discussed briefly.

Nomenclature

$A\!\!R$ = aspect ratio
c = semichord = 1
C_p = lifting pressure coefficient = $(p_- - p_+)/\frac{1}{2}\rho U^2$
$E(s,a)$ = error-index parameter [see Eq. (22)]
K = kernel function
k = reduced frequency = $\omega c/U$
NC = number of chordwise loading points
NS = total (full span) number of spanwise loading points
NQ = NC for reference solution
MQ = NS for reference solution
s = semispan
U = freestream velocity
w = downwash on wing
$x,y,z,$
ξ, η, ζ = Cartesian coordinates
x_0 = $x-\xi$
y_0 = $y-\eta$
θ = $\cos^{-1}(-y/s)$
φ = $\cos^{-1}(-x)$
ρ = air density

Subscripts and Superscripts

R = real part
I = imaginary part
a = approximate numerical solution
s = reference solution
$(\bar{\ })$ = complex amplitude

Introduction

THE evaluation of the pressure distribution on a wing is of prime importance for the design of airplanes. Many methods have been developed for calculating the pressure distribution on a thin finite wing in subsonic flow. The methods can be classified into two principal categories: the mode function method and the discrete element method. In the conventional discrete method, a lifting surface is divided into a number of small lifting elements (boxes). Typical procedures belonging to the discrete element method are the vortex lattice method[1] (VLM) and the doublet lattice method[2] (DLM). These methods have offered greater versatility for practical applications to complicated configurations.

The conventional VLM, however, has a deficiency in that the convergence of solutions is slow with respect to the number of elements used.[3] To improve the convergence, many practical considerations have been made so far, for example, the tip inset,[4] the semicircle method, and the sine law spacing. In 1974, Lan[5] presented an excellent numerical method, the quasi-vortex-lattice method, which gives exact results for two-dimensional airfoils and gives improved results for wings in steady flow. It seems that many deficiencies of the VLM were overcome by Lan's method.

In the unsteady case, where the DLM should be used instead of the VLM, the poor convergence of the DLM becomes serious, in particular, for high reduced frequency. Jordan[6] discussed the sources of error and presented an accurate method of evaluating influence function in the DLM formulation. The authors have presented an excellent method[7] for unsteady two-dimensional wings. In that case, the DLM error is mainly due to logarithmic terms in the kernel function. Through careful treatment of the terms, one can improve the DLM very much.

The purpose of the present investigation is to improve the DLM for three-dimensional wings. A key point of the present work is to use an expansion series of the kernel[8] together with proper treatment of the chordwise logarithmic singularity.[7] Unlike the conventional DLM, where the wing is divided into many boxes, the present method uses many chordwise strips. The expansion series of the kernel is used in the strip containing a control point, and spanwise integration is performed first to get a chordwise logarithmic singularity. The authors' previous work[7] is used to treat the chordwise logarithmic singularity accurately. Thus the present method gives excellent results over a wide range of k (k = 0.0-5.0) within reasonable computer time.

Doublet Strip Method

In thin wing theory for incompressible flow, the downwash on the wing is related to the pressure distribution through the following equations:

$$\frac{\bar{w}}{U}(x,y) = \frac{1}{8\pi} \int_{-s}^{s} \int_{-1}^{1} K(x_0,y_0;k)\, \bar{C}_p(\xi,\eta)\, d\xi d\eta \tag{1}$$

$$K(x_0,y_0;k) = e^{-ikx_0} \int_{-\infty}^{x_0} \frac{e^{ikv}}{(v^2+y_0^2)^{3/2}}\, dv \tag{2a}$$

$$K(x_0,y_0;k) = e^{-ikx_0} \frac{1}{y_0^2} \int_{-\infty}^{x_0/|y_0|} \frac{e^{ik|y_0|\lambda}}{(\lambda^2+1)^{3/2}}\, d\lambda \tag{2b}$$

where the coordinate system is illustrated in Fig. 1. In the conventional DLM, the lifting surface is divided into a number of small boxes. The wing in the present method, however, is divided into many chordwise strips according to the semicircle method (SCM). The strip which has the control point is called the singular strip (SS) and the other strips are called regular strips (RS) (see Fig. 1).

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*Research Associate, Department of Aeronautical Engineering. Member AIAA.

†Professor, Department of Aeronautical Engineering. Member AIAA.

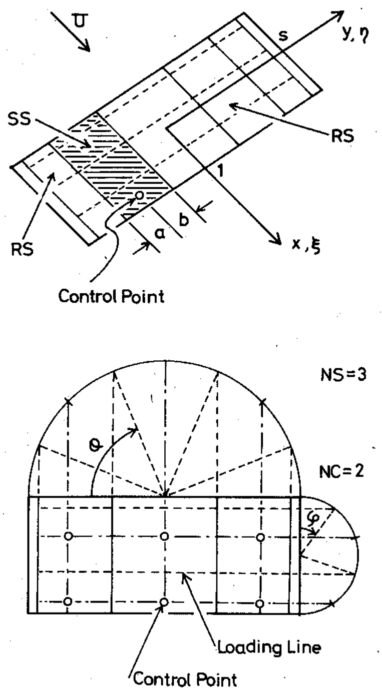

Fig. 1 Coordinates, SS and RS, SCM scheme.

It is assumed that the continuous pressure distribution over the wing is replaced with the one that is constant in the spanwise direction on each strip. Then, Eq. (1) can be written as

$$\left(\frac{\bar{w}}{U}\right)_{kl}=\frac{1}{8\pi}\sum_{i=1}^{NS}{}'\int_{d_i}^{d_{i+1}}\int_{-1}^{1}K(x_0,y_0;k)\,\bar{C}_p(\xi,\eta_i)\,\mathrm{d}\xi\mathrm{d}\eta$$
$$+\frac{1}{8\pi}\int_{y_k-a}^{y_k+b}\int_{-1}^{1}K(x_0,y_0;k)\,\bar{C}_p(\xi,y_k)\,\mathrm{d}\xi\mathrm{d}\eta \qquad (3)$$

(Σ' implies the exception of the singular strip), where

$$d_i=-s\cos[(2i-1)\pi/2(NS+1)]$$
$$\eta_i(y_k)=-s\cos[i(k)\pi/(NS+1)]$$
$$i,k=1,2,\ldots,NS \qquad (4)$$

Formulation for the Singular Strip

The kernel function K has often been evaluated approximately owing to difficulty in integrating Eq. (2). It seems, however, that the conventional approximations for K in the SS lead to unsatisfactory results because the kernel function has several singularities.

In this paper, an expansion series of the kernel[8] in the SS is used and spanwise integration is performed first. As seen later, this scheme has the following advantages: the kernel function can be evaluated more accurately; and all singularities are reduced to chordwise Cauchy and logarithmic singularities. The former singularity can be treated with Stark's formula[9] and the latter singularity with the method of Ref. 7.

The result is as follows:

$$\int_{y_k-a}^{y_k+b}K(x_0,y_0;k)\,\mathrm{d}\eta$$
$$=\int_{-b}^{a}K(x_0,y_0;k)\,\mathrm{d}y_0=e^{-ikx_0}(\hat{B}_R+i\hat{B}_I) \qquad (5)$$

$$\hat{B}_R=\sum_{n=0}^{\infty}(-1)^n\hat{U}_{2n}-\frac{k^2}{2}\sum_{n=0}^{\infty}\frac{1}{(n+1)(n!)^2}\left(\sum_{m=1}^{n}\frac{1}{m}\right.$$
$$\left.+\frac{1}{2(n+1)}-\gamma-\ln\frac{k}{2}\right)\left(\frac{k}{2}\right)^{2n}\frac{a^{2n+1}+b^{2n+1}}{2n+1} \qquad (6a)$$

$$\hat{B}_I=\sum_{n=0}^{\infty}(-1)^n\hat{U}_{2n+1}$$
$$+\frac{\pi}{4}k^2\sum_{n=0}^{\infty}\frac{1}{(n+1)(n!)^2}\left(\frac{k}{2}\right)^{2n}\frac{a^{2n+1}+b^{2n+1}}{2n+1} \qquad (6b)$$

$$\hat{U}_m=\int_{-b}^{a}U_m\mathrm{d}y_0 \qquad (7)$$

Several leading terms of Eq. (7) can be given analytically as

$$\hat{U}_0=-1/a-1/b-\sqrt{x_0^2+a^2}/ax_0-\sqrt{x_0^2+b^2}/bx_0 \qquad (8a)$$

$$\hat{U}_1=-k[\ln(\sqrt{x_0^2+a^2}+a)+\ln(\sqrt{x_0^2+b^2}+b)-2\ln|x_0|] \qquad (8b)$$

$$\hat{U}_2=-(k^2/2)\,[a\ln(\sqrt{x_0^2+a^2}-x_0)$$
$$+b\ln(\sqrt{x_0^2+b^2}-x_0)-a-b] \qquad (8c)$$

$$\hat{U}_3=(k^3/6)\,(a\sqrt{x_0^2+a^2}+b\sqrt{x_0^2+b^2}) \qquad (8d)$$

$$\hat{U}_4=(k^4x_0/48)\,(a\sqrt{x_0^2+a^2}+b\sqrt{x_0^2+b^2})$$
$$+(k^4/48)\,[a^3\ln(\sqrt{x_0^2+a^2}-x_0)$$
$$+b^3\ln(\sqrt{x_0^2+b^2}-x_0)]-(k^4/144)\,(a^3+b^3) \qquad (8e)$$

$$\hat{U}_5=(k^5/360)\,[(ax_0^2-2a^3)\sqrt{x_0^2+a^2}$$
$$+(bx_0^2-2b^3)\sqrt{x_0^2+b^2}] \qquad (8f)$$

U_m $(m>5)$ is given by numerical integration for U_m (U_m is given in Ref. 8). As Eq. (8b) has a chordwise logarithmic singularity, it is possible to write $\hat{B}_I=\tilde{B}_I+2k\ln|x_0|$. Thus integration on the SS [the second integral of Eq. (3)] can be written as

$$(1/8\pi)\int_{-1}^{1}\bar{C}_p(\xi,y_k)\,e^{-ikx_0}\,[\hat{B}_R+i\tilde{B}_I+2ik\ln|x_0|]\,\mathrm{d}\xi$$
$$=(1/8\pi)\,[I_R+2ike^{-ikx}I_{\ln}] \qquad (9)$$

$$I_R=\int_{-1}^{1}e^{-ikx_0}(\hat{B}_R+i\tilde{B}_I)\,\bar{C}_p(\xi,y_k)\,\mathrm{d}\xi \qquad (10)$$

$$I_{\ln}=\int_{-1}^{1}e^{ik\xi}\ln|x-\xi|\,\bar{C}_p(\xi,y_k)\,\mathrm{d}\xi \qquad (11)$$

Since $I_{\ln}$ is the same form as the term appearing in Ref. 7, it can be discretized accurately in the same manner. (The derivation is described in the Appendix of this paper.) The result is

$$I_{\ln}=\frac{2\pi}{2NC+1}\sum_{j=1}^{NC}K_{\ln}e^{-ik\cos\varphi_j}\tan\frac{\varphi_j}{2}\,\bar{C}_p(\xi_j,y_k) \qquad (12)$$

$$K_{\ln}=\sum_{n=0}^{NC-1}\left[\cos n\varphi_j+\cos(n+1)\varphi_j\right]F_n(\varphi_l) \qquad (13a)$$

$$F_n(\varphi_l)=-(\ln 2+\cos\varphi_l) \qquad n=0$$
$$=-\left[\frac{\cos n\varphi_l}{n}+\frac{\cos(n+1)\varphi_l}{n+1}\right] \qquad n\geq 1 \qquad (13b)$$

where

$$\xi_j(x_l) = -\cos\varphi_j(\varphi_l) \qquad \varphi_j = (2j-1)\pi/(2NC+1)$$
$$\varphi_l = 2l\pi/(2NC+1)$$
$$j,l = 1,2,\ldots,NC \tag{14}$$

On the other hand, the remaining term I_R can be reduced to a finite sum through the Stark[9] and Gauss[10] formula because the integrand has a Cauchy singularity and other regular terms; thus

$$I_R = \frac{2\pi}{2NC+1}\sum_{j=1}^{NC} K_R \bar{C}_p(\xi_j, y_k)\sin\varphi_j \tag{15}$$

$$K_R = e^{-ikx_0}[\hat{B}_R + i\tilde{B}_I] \tag{16}$$

The discretizing method of I_R is similar to that of Lan,[5] but the two methods are different in their layout of the loading and control points. In order to get the same layout of the points as for the term $I_{\ell n}$, the Stark and Gauss formula is used. Substitution of Eqs. (12) and (15) into Eq. (9) yields

$$\frac{1}{8\pi}\int_{y_k-a}^{y_k+b}\int_{-1}^{1} K(x_0,y_0;k)\bar{C}_p(\xi,y_k)\,\mathrm{d}\xi\mathrm{d}\eta = \sum_{j=1}^{NC} K_j^l \bar{C}_{pkj} \tag{17}$$

$$K_j^l = \frac{1}{4(2NC+1)}\left[K_R\sin\varphi_j + 2ike^{-ikx_0}K_{\ell n}\tan\frac{\varphi_j}{2}\right] \tag{18}$$

Formulation for the Regular Strip

The treatment of the integral is straightforward since the integrand in the RS contains no singularity. Using the Gauss formula, the first integral of Eq. (3) is discretized to be

$$\frac{1}{8\pi}\sum_{i=1}^{NS}{}'\int_{d_i}^{d_{i+1}}\int_{-1}^{1} K(x_0,y_0;k)\bar{C}_p(\xi,\eta_i)\,\mathrm{d}\xi\mathrm{d}\eta$$
$$= \sum_{i=1}^{NS}{}'\sum_{j=1}^{NC} D_{ij}^{kl}\bar{C}_{pij} \tag{19}$$

$$D_{ij}^{kl} = \frac{\sin\varphi_j}{4(2NC+1)}\int_{d_i}^{d_{i+1}} K(x_l-\xi_j, y_k-\eta;k)\,\mathrm{d}\eta \tag{20}$$

where the η-wise integration for the kernel [Eq. (20)] is performed by the same method as in Ref. 2 which uses a parabolic approximation for K ($\mathrm{K}=K\times y_0^2$).

Formulation for the Singular Strip Plus the Regular Strip

Combining Eqs. (17) and (19) one obtains

$$(\bar{w}/U)_{kl} = \sum_{i=1}^{NS}\sum_{j=1}^{NC}[(1-\delta_{ik})D_{ij}^{kl} + \delta_{ik}K_j^l]\bar{C}_{pij} \tag{21}$$

where δ_{ik} is the Kronecker delta. As shown later, the present method gives excellent results up to $k \sim 5.0$ within reasonable computer time.

Error Estimation

In order to estimate the accuracy of the lift distribution, an error-index parameter is defined as follows:

$$E(s,a) = \frac{\sum_{i=1}^{NS}\sum_{j=1}^{NC}\left|\Delta\bar{C}_p^R\right|h_{ij} + \sum_{i=1}^{NS}\sum_{j=1}^{NC}\left|\Delta\bar{C}_p^I\right|h_{ij}}{\sum_{i=1}^{MQ}\sum_{j=1}^{NQ}\left|\bar{C}_{ps}^R\right|h_{ij} + \sum_{i=1}^{MQ}\sum_{j=1}^{NQ}\left|\bar{C}_{ps}^I\right|h_{ij}} \tag{22}$$

$$\Delta\bar{C}_p = \bar{C}_{pa} - \bar{C}_{ps} \qquad h_{ij} = \sin\theta_i\sin\varphi_j \tag{23}$$

When the exact solution is available, it can be chosen as the reference value C_{ps}. Unfortunately, there is no exact solution for unsteady rectangular wings. Thus the present method itself with a large number of loading points may be regarded as the reference solution C_{ps} since the convergence of solution is very rapid, as shown in Figs. 2 and 3.

In the calculation of $E(s,a)$, $NQ=18$ and $MQ=35$ for the $\mathcal{R}=2$ wing and $NQ=15$ and $MQ=45$ for the $\mathcal{R}=4$ wing are chosen as the reference solutions. Intermediate values are obtained using the interpolation functions[11] because the locations of the loading points of an approximate method do not always coincide with those of reference solutions.

Numerical Results

For comparison, we chose the DLM[2] which is formulated by using equispaced lattice without the tip inset (designated DLM-EQ in this paper), the Davies method,[11] and the doublet lattice method-semicircle method (DLM-SC). The last method is similar to the present method except that the SS is treated in

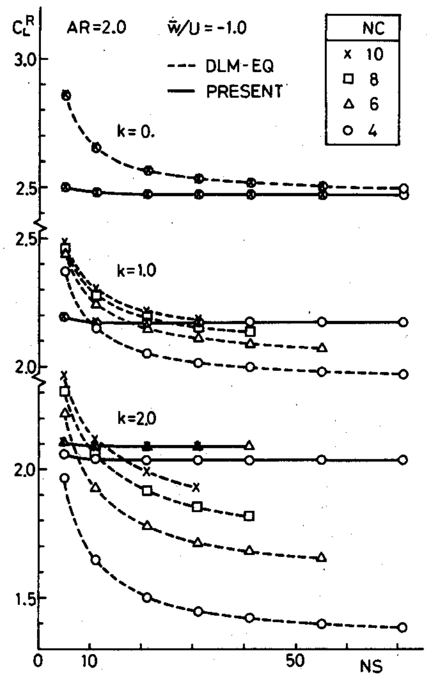

Fig. 2 Effect of varying the number of loading points on C_L (real part).

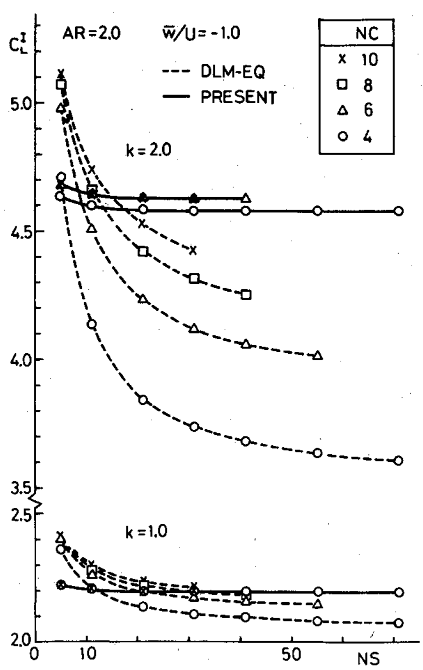

Fig. 3 Effect of varying the number of loading points on C_L (imaginary part).

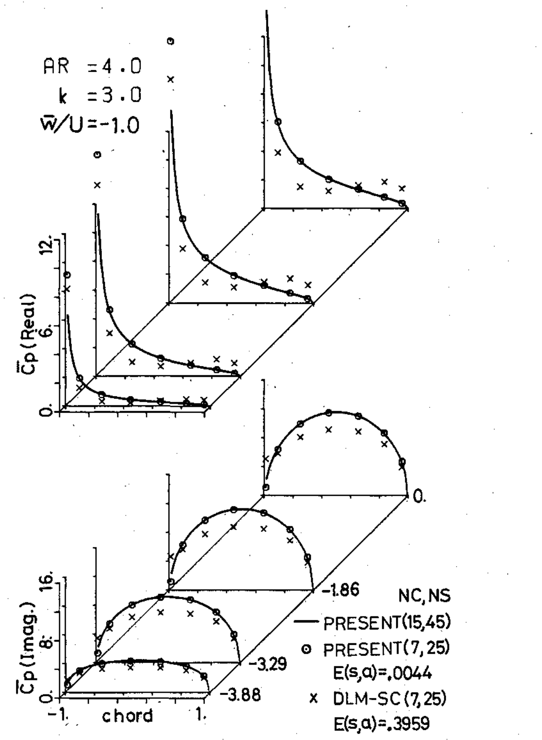

Fig. 4 An example of pressure distribution for a rectangular wing.

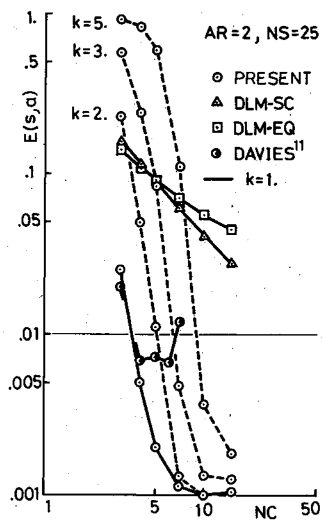

Fig. 5 Comparison of $E(s,a)$ with NC.

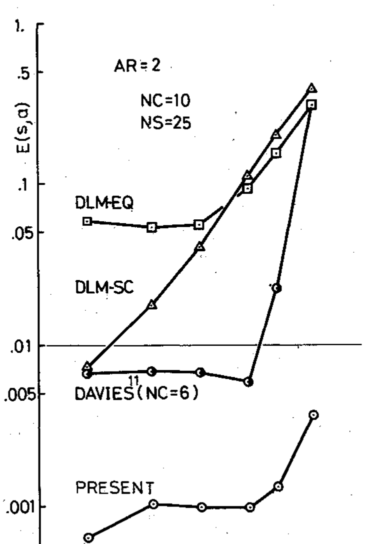

Fig. 6 Comparison of $E(s,a)$ with k.

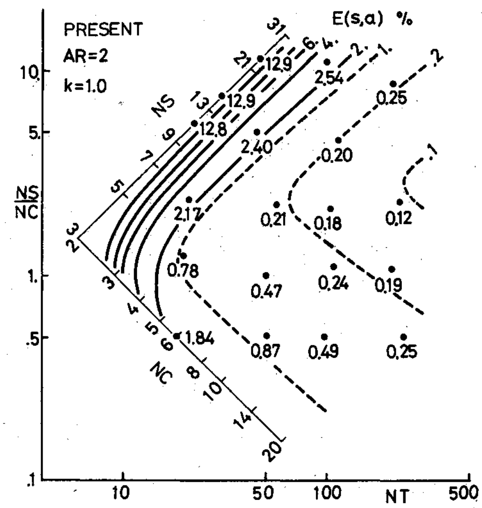

Fig. 7 $E(s,a)$ contour for present method.

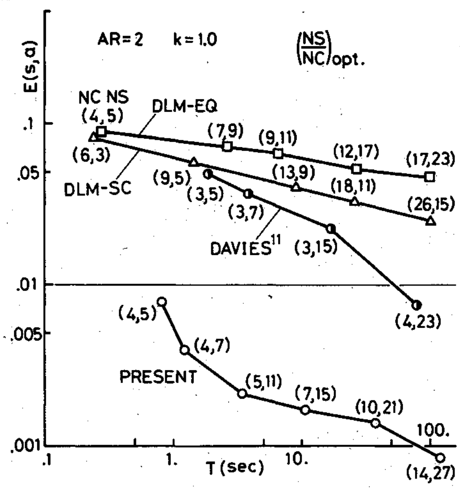

Fig. 8 Comparison of $E(s,a)$ with T.

the same manner as the RS. Comparing results of the present method with those of the DLM-SC, the importance of the present treatment for the SS is apparent. When reduced frequency is zero, the DLM-SC gives the result as good as the Lan method.[5] In the following numerical examples, it is assumed that the wing planform is rectangular with $\mathcal{R} = 2$ or 4, downwash on the wing is $\bar{w}/U = -1$, and the number of unknowns has been reduced by about one-half owing to the symmetry of wings.

Figures 2 and 3 compare C_L convergence properties of the present method with those of the DLM-EQ. It is to be noted that the present method improves the convergence remarkably. As is shown, the poor convergence of the DLM-EQ becomes serious with increasing reduced frequency. Though the DLM with the tip inset or with the semicircle method is used, the convergence of the method is improved only for relatively low reduced frequency. On the other hand, the convergence of the present method is very rapid even for high reduced frequency. It can be seen that the present method, with $NC = 6$ and $NS = 11$, results in a converged solution at $k = 2.0$.

An example of the pressure distribution on a rectangular wing with $\mathcal{R} = 4$ is shown in Fig. 4, where only the left-hand side of the wing is illustrated. It should be noted that the present solutions with $NC = 7$ and $NS = 25$ almost converge to the reference solutions (solid lines).

Figure 5 shows $E(s,a)$ vs NC, with $NS = 25$ fixed for the present method and other three methods. It is found that the present method is superior to other methods. The Davies method seems to be good, but the method has a defect: too large a value of NC reduces the accuracy of solution.

Figure 6 shows $E(s,a)$ vs k. The DLM-EQ has $E(s,a) > 0.1$ above $k = 2.0$ and is not appropriate. But the present method gives excellent results even at $k = 5.0$. Although more terms in the expansion series of the kernel function are required with increasing k, the increase in computer time is not serious.

Figure 7 shows $E(s,a)$ contours on the NS/NC vs NT ($= NC \times NS$) plane for the present method. From this figure, one can estimate the optimum NS/NC when NT is prescribed and can find the convergence property when NC and/or NS are increased.

Figure 8 shows $E(s,a)$ vs T when $(NS/NC)_{opt}$ for each method is selected. Namely, this figure indicates the efficiency of each method. The present method obtains $E(s,a) = 0.01$ (accurate enough for practical use) with a computer time of about 0.8 s (we use a FACOM M-200 computer in Nagoya University), while the Davies method requires about 50 s. The DLM-EQ and the DLM-SC methods require far more computer time.

Conclusion

An improved numerical method (doublet strip method) for calculating lift distributions on unsteady rectangular wings has been developed. The kernel function is treated in the form of the expansion series in the singular strip, and spanwise integration is performed first to get an explicit expression of the chordwise logarithmic singularity. This singularity can be treated very accurately by the authors' previous method.

The present method gives excellent results even for high reduced frequency of as much as $k = 5.0$. At such frequencies, the conventional DLM is not applicable. The computer time of the present method is comparable with that of the conventional DLM: thus, if a prescribed accuracy of solution is desired, the present method needs much less computer time than those of many other current methods.

It remains to generalize the method to include compressibility, swept tapered wings, etc. For rectangular wings in compressible flow, the formulation was given and reported.[12] For swept tapered wings, it seems that the transformation into rectangular domains is needed.

The authors found, after they fully developed the method, that a method to treat the logarithmic terms in the chordwise integration had been described by Lan.[13] Though his method is different from that of the authors, comparison between two methods may be required in future investigations.

Appendix

The term $e^{ik\xi}\bar{C}_p(\xi, y_k)$ can be written as follows:

$$e^{ik\xi}\bar{C}_p(\xi, y_k) = \sqrt{(1-\xi)/(1+\xi)}\, f(\xi) \tag{A1}$$

where $f(\xi)$ is a well-behaved function. Substituting Eq. (A1) into Eq. (11) and putting $x = -\cos\varphi_l$ and $\xi = -\cos\varphi$, one obtains

$$I_{\ell n} = \int_0^\pi (1+\cos\varphi)\,\ell n\,|\cos\varphi_l - \cos\varphi|\, f(\varphi)\, d\varphi \tag{A2}$$

Now, we use the interpolated expression for $f(\varphi)$:

$$f(\varphi) = \sum_{j=1}^{NC} g_j(\varphi) f_j \qquad f_j \equiv f(\varphi_j) \tag{A3}$$

where $g_j(\varphi)$ is the interpolation function[11] based on the weighting function $\sqrt{(1-\xi)/(1+\xi)}$:

$$g_j(\varphi) = \frac{4}{2NC+1}\,\frac{\cos\varphi_j/2}{\cos\varphi/2} \sum_{n=0}^{NC-1} \cos(n+1/2)\varphi_j \cos(n+1/2)\varphi$$

$$\varphi_j = (2j-1)\pi/(2NC+1) \qquad j = 1,2,\ldots,NC \tag{A4}$$

Substitution of Eqs. (A3) and (A4) into Eq. (A2) requires evaluation of the integral

$$I_{\ell n} = \int_0^\pi \cos n\varphi\,\ell n\,|\cos\varphi_l - \cos\varphi|\, d\varphi \tag{A5a}$$

$$\begin{aligned} I_{\ell n} &= -\pi\ell n 2 & n&=0 \\ &= -(\pi/n)\cos n\varphi_l & n&\geq 1 \end{aligned} \tag{A5b}$$

which is obtained by using the formula

$$\ell n\,|\cos\varphi_l - \cos\varphi| = -\ell n 2 - 2\sum_{r=1}^{\infty} \frac{1}{r}\cos r\varphi_l \cos r\varphi \tag{A6}$$

Thus we have

$$I_{\ell n} = \frac{2\pi}{2NC+1} \sum_{j=1}^{NC} K_{\ell n} f_j$$

$$K_{\ell n} = \sum_{n=0}^{NC-1} [\cos n\varphi_j + \cos(n+1)\varphi_j] Fn(\varphi_l)$$

$$\begin{aligned} Fn(\varphi_l) &= -(\ell n 2 + \cos\varphi_l) & n&=0 \\ &= -\left[\frac{\cos n\varphi_l}{n} + \frac{\cos(n+1)\varphi_l}{n+1}\right] & n&\geq 1 \end{aligned} \tag{A7}$$

By using $f_j = e^{ik\xi_j}\tan(\varphi_j/2)\bar{C}_p(\xi_j, y_k)$, one can get the expressions Eqs. (12) and (13) in the text.

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